

\mathbb{Z}_n^3 -graded colored supersymmetry

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Abstract

We build generalizations of the Grassmann algebras from a few simple assumptions which are that they are graded, maximally symmetric and contain an ordinary Grassmann algebra as a subalgebra. These algebras are graded by \mathbb{Z}_n^3 and display surprising properties that indicate their possible application to the modelization of quark fields. We build the generalized supersymmetry generators based on these algebras and their derivation operators. These generators are *cubic* roots of the usual supersymmetry generators.

1 Introduction

Many attempts[1, 2, 3, 8] have been made to replace the \mathbb{Z}_2 grading group that is usually used in supersymmetry[4, 5] by another abelian group. Most of these attempts deal with $(\mathbb{Z}_2)^n$ grading groups and the corresponding generalized Lie algebras, that are called Color Lie Superalgebras. It has been shown by Scheunert[1] that any ε -Lie superalgebra, of which the Color Lie Superalgebras are particular cases, is isomorphic to an ordinary Lie superalgebra, leaving little room for a generalization of supersymmetry based on an alternative grading group, at least at the level of the Lie superalgebra. Attempts have also been made to generalize supersymmetry by replacing the Grassmann algebra with a paragrassmann algebra, but it is not clear what the algebraic structures corresponding to Lie superalgebras are in this case, and these algebras are very unsymmetrical: they need an ordering relation between the generators[6, 7]. We build a generalization of the Grassmann algebras from different assumptions and

build the corresponding generalized supersymmetry generators. The resulting algebraic structure is an ε -Lie superalgebra (the Jacobi identity is satisfied), and it has a subalgebra that is a Lie superalgebra.

2 Generalized Grassmann algebras

Any grading group \mathcal{G} , being commutative, can be decomposed into the product of \mathbb{Z} or \mathbb{Z}_{p_i} groups

$$\mathcal{G} = \prod_{i=1}^n \mathbb{Z}_{p_i}$$

where p_i is 0 by convention for a \mathbb{Z} group. If an element A of a \mathcal{G} -graded structure is of grade a , we will note $a = \{a_i; 1 \leq i \leq n\}$ to distinguish the integers characterizing the grade of A in each of the \mathbb{Z}_{p_i} groups. One can then define a \mathcal{G} -graded commutation factor, whose general form[8, 1] is

$$\varepsilon(a, b) = \prod_{i=1}^n (\pm 1)^{a_i b_i} \prod_{1 \leq i < j \leq n} q_{ij}^{a_i b_j - a_j b_i}$$

where the q_{ij} are r_{ij}^{th} roots of unity, r_{ij} being the greatest common divider of p_i and p_j . One can choose -1 in the first factors only in the case where p_i is even.

The assumptions we make about our generalized Grassmann algebra are that:

- it is ε -commutative, that is

$$AB = \varepsilon(g_A, g_B)BA$$

for all A and B in the algebra, where g_A and g_B are the grades of A and B .

- the \mathbb{Z}_{p_i} groups are all equivalent and undistinguishable.
- it contains a fermionic sector (formed with elements that anticommute with each other) that put on an equal footing the \mathbb{Z}_{p_i} groups.

Any Grassmann algebra satisfying these assumptions with a grading group composed of the product of two \mathbb{Z}_{p_i} groups is isomorphic to an ordinary Grassmann algebra. No Grassmann algebra with a grading group composed of the product of four or more \mathbb{Z}_{p_i} groups is able to put them on an equal footing, unless it is isomorphic to an ordinary Grassmann algebra. This is because it is impossible to build a symmetric oriented graph with four points. Here are the four possible oriented graphs with four points:

If we allow for different commutation factors between different grading groups, then the *pairs* of groups become distinguishable. On the other hand, with three points, it is possible to build a symmetric oriented graph. Here are the two possible oriented graphs with three points:

Note that the paragrassmann algebras with three and four generators are described by the first triangle and tetrahedron. The second triangle is perfectly symmetric in the three points and can be used to build our generalized Grassmann algebra. We chose here to label each \mathbb{Z}_n group with a color, red, green or blue and to note a_R , a_G and a_B the components in the red, green and blue \mathbb{Z}_n groups of an element a of the grading group. The corresponding commutation factor is

$$\begin{aligned}\epsilon(a, b) &= (-1)^{a_R b_R + a_G b_G + a_B b_B} (q)^{a_R b_G - a_G b_R + a_G b_B - a_B b_G + a_B b_R - a_R b_B} = \\ &= (-1)^{a_R b_R + a_G b_G + a_B b_B} (q)^{a_R b_G + a_G b_B + a_B b_R} (q^{-1})^{a_R b_B + a_B b_G + a_G b_R}\end{aligned}$$

and the smallest grading group is $\mathbb{Z}_n^3 = \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ if q is an n^{th} root of 1 with n even, \mathbb{Z}_{2n}^3 if q is an n^{th} root of 1 with n odd, or \mathbb{Z}^3 if q is not a root of 1 (in this case, we set $n = 0$).

Let us examine in detail the generators one can have in our algebra and their commutation relations. We can have monochromatic generators having only one

of their three colors equal to one. Then, one can have “anti-monochromatic” generators that have one of their colors equal to -1 . These are the conjugates of monochromatic generators. Then, one can have “black” generators with all their colors zero and “white” generators with their three colors all equal but different from zero. There are other combinations but we won’t concentrate on them for the moment. Let us denote by $\theta_{AR}, \theta_{AG}, \theta_{AB}$ the red, green and blue monochromatic generators, by $\bar{\theta}_{\bar{A}\bar{R}}, \bar{\theta}_{\bar{A}\bar{G}}, \bar{\theta}_{\bar{A}\bar{B}}$ the red, green and blue anti-monochromatic generators, by e_a the black generators and by η_α (resp. $\bar{\eta}_{\bar{\alpha}}$) the white (resp. anti-white, that is, with all colors equal to -1) generators. We summarize this in the following table:

	θ_R	θ_G	θ_B	$\bar{\theta}_{\bar{R}}$	$\bar{\theta}_{\bar{G}}$	$\bar{\theta}_{\bar{B}}$	e	η	$\bar{\eta}$
Red	1	0	0	-1	0	0	0	1	-1
Green	0	1	0	0	-1	0	0	1	-1
Blue	0	0	1	0	0	-1	0	1	-1

and take the convention that capitalized latin indices refer to monochromatic degrees of freedom, small latin indices to black degrees of freedom and greek indices to white degrees of freedom.

First, one can note that our wish to have a symmetric fermionic sector is fulfilled: the black and white sectors form a subalgebra that is an usual Grassmann algebra:

$$\eta_\alpha \eta_\beta = -\eta_\beta \eta_\alpha, \quad \bar{\eta}_{\bar{\alpha}} \bar{\eta}_{\bar{\beta}} = -\bar{\eta}_{\bar{\beta}} \bar{\eta}_{\bar{\alpha}}, \quad \eta_\alpha \bar{\eta}_{\bar{\beta}} = -\bar{\eta}_{\bar{\beta}} \eta_\alpha$$

$$e_a e_b = e_b e_a, \quad e_a \eta_\alpha = \eta_\alpha e_a, \quad e_a \bar{\eta}_{\bar{\alpha}} = \bar{\eta}_{\bar{\alpha}} e_a$$

An important remark is that the only combinations of monochromatic generators that can be included in this Grassmann subalgebra are the $\theta_R \theta_G \theta_B$ (and all similar terms with the three colors in a different order), the $\bar{\theta}_{\bar{R}} \bar{\theta}_{\bar{G}} \bar{\theta}_{\bar{B}}$, the $\theta_R \bar{\theta}_{\bar{R}}$, $\theta_G \bar{\theta}_{\bar{G}}$, $\theta_B \bar{\theta}_{\bar{B}}$, the product of n monochromatic generators of the same color, and their products. Any other combination destroys the \mathbb{Z}_2 -grading if the colors are maintained on an equal footing. That is, if we include θ_R , we must also include θ_G and θ_B , which destroys the \mathbb{Z}_2 -grading because $\theta_R \theta_G = q \theta_G \theta_R$. These combinations bear a very strong formal resemblance with the only physically observable combinations of quarks in QCD, especially for $n = 0$. But where QCD gives a dynamical explanation for the confinement, our model would derive the rules of confinement from algebraical and statistical arguments: a Fock space generated by operators obeying our commutation rules would have some states that, because of their colored content, would not be entirely symmetric or antisymmetric and so would be physically unobservable. On the other hand, states containing only black or white combinations of monochromatic generators would have correct symmetry properties with respect to the exchange of the physically observable particles, that is hadrons. Please note that we have not imposed any condition on the gauge theory but only on the grading group. But

because of the assumed symmetry between the three colors, $SU(3)$ and QCD seem very natural in this framework.

The meson-like combinations of monochromatic generators, the $\theta_R \bar{\theta}_{\bar{R}}$, $\theta_G \bar{\theta}_{\bar{G}}$, and $\theta_B \bar{\theta}_{\bar{B}}$, commuting with the rest of the algebra, are *bosons*. The baryon-like combinations, $\theta_R \theta_G \theta_B$ and $\bar{\theta}_{\bar{R}} \bar{\theta}_{\bar{G}} \bar{\theta}_{\bar{B}}$, anticommuting with each other, with the white generators η and $\bar{\eta}$ and their odd combinations, are *fermions*. Individually, the monochromatic generators commute with the even sector of the algebra (that is the bosons) and anticommute with the odd sector (that is the fermions) and so from our black and white world look like fermions, but to each other, they do not act like fermions, but rather like parafermions:

$$\begin{aligned}\theta_R \theta_G &= q \theta_G \theta_R, & \theta_G \theta_B &= q \theta_B \theta_G, & \theta_B \theta_R &= q \theta_R \theta_B \\ \bar{\theta}_{\bar{R}} \bar{\theta}_{\bar{G}} &= q \bar{\theta}_{\bar{G}} \bar{\theta}_{\bar{R}}, & \bar{\theta}_{\bar{G}} \bar{\theta}_{\bar{B}} &= q \bar{\theta}_{\bar{B}} \bar{\theta}_{\bar{G}}, & \bar{\theta}_{\bar{B}} \bar{\theta}_{\bar{R}} &= q \bar{\theta}_{\bar{R}} \bar{\theta}_{\bar{B}}\end{aligned}$$

with important differences, among which $(\theta_i)^2 = \theta_i \theta_i = -\theta_i \theta_i = 0$, which indicates that, as fermions, two monochromatic particles would be unable to be in the same state at the same point. From these relations, we easily show that the θ 's obey the ternary rule $\theta_R \theta_G \theta_B = \theta_G \theta_B \theta_R = \theta_B \theta_R \theta_G$ and not $\theta_R \theta_G \theta_B = j \theta_G \theta_B \theta_R = j^2 \theta_B \theta_R \theta_G$ which was the fundamental relation of the generalized Grassmann algebras Kerner introduced in [9].

3 Derivations of the generalized Grassmann algebra

Like one built the supersymmetry generators with the ordinary Grassmann algebra, one can build with this tricolor Grassmann algebra the generators of a tricolor supersymmetry. First, let us define the derivation operators on the algebra of tricolor Grassmann valued functions from the twisted Leibniz rule:

$$[\partial^x, X_y]_c = \delta_y^x \mathbf{1}$$

where, by x and y we mean any kind of index, colored or not, and by X , the multiplication by any generator of the tricolor Grassmann algebra, colored or not. The colored commutator is defined by

$$[A, B]_c = AB - \varepsilon(g_A, g_B) BA$$

where g_A and g_B are the grades of A and B . The red (resp. green, blue) derivation operators are attributed the red (resp. green, blue) grade -1 . The ∂^a are black, the ∂^α have all their colors equal to -1 and the $\bar{\partial}^{\bar{\alpha}}$ have all their colors equal to 1 . Thus, the derivations on black and white indices obey the usual superleibniz rule.

From these rules, we can derive for example

$$\partial^{A_R}(\theta_{C_G} f) = q^{-1} \theta_{C_G} \partial^{A_R} f, \quad \partial^{A_R}(\theta_{C_R} \theta_{D_R}) = \delta_{C_R}^{A_R} \theta_{D_R} - \delta_{D_R}^{A_R} \theta_{C_R}, \quad \partial^{A_R}(\theta_{A_R} f) = f - \theta_{A_R} \partial^{A_R} f$$

The colors the derivation operators have been assigned are consistent with the fact that the colored commutator of any pair of derivation operators vanishes. For example,

$$[\partial^R, \partial^G]_c = \partial^R \partial^G - q \partial^G \partial^R = 0$$

In other words, the derivations have the *same* commutation relations as the corresponding generators.

4 The colored Supersymmetry generators

Let us redefine first the usual supersymmetry generators:

$$D^\alpha = \partial^\alpha + \sigma_a^{\alpha\dot{\alpha}} \bar{\eta}_{\dot{\alpha}} \partial^a \quad \text{and} \quad \bar{D}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} + \sigma_a^{\alpha\dot{\alpha}} \eta_\alpha \partial^a$$

These operators are exactly identical to the usual supersymmetry generators and have the same properties, in particular

$$[D^\alpha, \bar{D}^{\dot{\alpha}}]_c = \{D^\alpha, \bar{D}^{\dot{\alpha}}\} = 2\sigma_a^{\alpha\dot{\alpha}} \partial^a$$

that is, the supersymmetric translations are “square roots” of the translations of the commutative sector. Similar, but richer properties can be derived for the tricolor supersymmetry generators:

$$\begin{aligned} D^{A_R} &= \partial^{A_R} + \rho_\alpha^{A_R C_G D_B} \theta_{C_G} \theta_{D_B} D^\alpha + \omega_a^{A_R \bar{A}_R} \bar{\theta}_{\bar{A}_R} \partial^a \\ D^{C_G} &= \partial^{C_G} + \rho_\alpha^{A_R C_G D_B} \theta_{D_B} \theta_{A_R} D^\alpha + \omega_a^{C_G \bar{C}_G} \bar{\theta}_{\bar{C}_G} \partial^a \\ D^{D_B} &= \partial^{D_B} + \rho_\alpha^{A_R C_G D_B} \theta_{A_R} \theta_{C_G} D^\alpha + \omega_a^{D_B \bar{D}_B} \bar{\theta}_{\bar{D}_B} \partial^a \end{aligned}$$

and their conjugates

$$\begin{aligned} \bar{D}^{\bar{A}_R} &= \bar{\partial}^{\bar{A}_R} + \bar{\rho}_{\dot{\alpha}}^{\bar{A}_R \bar{C}_G \bar{D}_B} \bar{\theta}_{\bar{C}_G} \bar{\theta}_{\bar{D}_B} \bar{D}^{\dot{\alpha}} + \bar{\omega}_a^{\bar{A}_R A_R} \theta_{A_R} \partial^a \\ \bar{D}^{\bar{C}_G} &= \bar{\partial}^{\bar{C}_G} + \bar{\rho}_{\dot{\alpha}}^{\bar{A}_R \bar{C}_G \bar{D}_B} \bar{\theta}_{\bar{D}_B} \bar{\theta}_{\bar{A}_R} \bar{D}^{\dot{\alpha}} + \bar{\omega}_a^{\bar{C}_G C_G} \theta_{C_G} \partial^a \\ \bar{D}^{\bar{D}_B} &= \bar{\partial}^{\bar{D}_B} + \bar{\rho}_{\dot{\alpha}}^{\bar{A}_R \bar{C}_G \bar{D}_B} \bar{\theta}_{\bar{A}_R} \bar{\theta}_{\bar{C}_G} \bar{D}^{\dot{\alpha}} + \bar{\omega}_a^{\bar{D}_B D_B} \theta_{D_B} \partial^a \end{aligned}$$

We also define the following alternative operators where the θ 's in the second term of the definition of the D operators have been swapped:

$$D'^{A_R} = \partial^{A_R} + \rho_\alpha^{A_R C_G D_B} \theta_{D_B} \theta_{C_G} D^\alpha + \omega_a^{A_R \bar{A}_R} \bar{\theta}_{\bar{A}_R} \partial^a$$

D'^{C_G} , D'^{D_B} , $\bar{D}'^{\bar{A}_R}$, $\bar{D}'^{\bar{C}_G}$ and $\bar{D}'^{\bar{D}_B}$ are defined similarly.

It is straightforward to show that

$$\begin{aligned} [D^{A_R}, [D'^{A_G}, D^{A_B}]_c]_c &= [D^{A_G}, [D'^{A_B}, D^{A_R}]_c]_c = [D^{A_B}, [D'^{A_R}, D^{A_G}]_c]_c = \\ &= (q-1) \rho_\alpha^{A_R A_G A_B} D^\alpha \end{aligned}$$

and similar relations with barred operators. That is, the monochromatic supersymmetric translations are also *cubic* roots of the ordinary *supersymmetric* translations.

5 The \mathbb{Z}_n^3 -graded ε -Lie superalgebra

The algebra generated by the D , \bar{D} and ∂ operators by means of the colored commutator is finite-dimensional (if the generalized Grassmann algebra is generated by a finite number of generators, of course). In fact, our colored supersymmetric translations generate an ε -Lie superalgebra[1], where the product is the colored commutator.

To write the commutation relations, we need to define the other elements of the algebra:

$$\begin{aligned} Q^{A_R A_G} &= \rho_\beta^{A_R A_G B_B} \theta_{B_B} D^\beta, \quad Q^{A_R \dot{\alpha}} = \rho_\beta^{A_R B_G B_B} \sigma_b^{\beta \dot{\alpha}} \theta_{B_G} \theta_{B_B} \partial^b, \quad Q^{A_R A_G \dot{\alpha}} = \rho_\beta^{A_R A_G B_B} \sigma_b^{\beta \dot{\alpha}} \theta_{B_B} \partial^b \\ Q^{A_R A_G \bar{A}_R \bar{A}_B} &= \rho_\beta^{A_R A_G B_B} \bar{\rho}_{\dot{\beta}}^{\bar{A}_R \bar{B}_G \bar{A}_B} \sigma_b^{\beta \dot{\beta}} \theta_{B_B} \bar{\theta}_{\bar{B}_G} \partial^b, \quad Q^{A_R A_G \bar{A}_R} = \rho_\beta^{A_R A_G B_B} \bar{\rho}_{\dot{\beta}}^{\bar{A}_R \bar{B}_G \bar{B}_B} \sigma_b^{\beta \dot{\beta}} \theta_{B_B} \bar{\theta}_{\bar{B}_G} \bar{\theta}_{\bar{B}_B} \partial^b \\ Q^{A_R \bar{A}_G} &= \rho_\beta^{A_R B_G B_B} \bar{\rho}_{\dot{\beta}}^{\bar{B}_R \bar{A}_G \bar{B}_B} \sigma_b^{\beta \dot{\beta}} \theta_{B_G} \theta_{B_B} \bar{\theta}_{\bar{B}_B} \partial^b \end{aligned}$$

and the similar operators in other colors, or conjugated. The commutation relations that haven't been given in the previous section are

$$\begin{aligned} [D^{A_R}, \bar{D}^{\bar{A}_R}]_c &= q[D'^{A_R}, \bar{D}^{\bar{A}_R}]_c = q[D^{A_R}, \bar{D}'^{\bar{A}_R}]_c = q^2[D'^{A_R}, \bar{D}'^{\bar{A}_R}]_c = \\ &= Q^{A_R \bar{A}_R} + (\omega_a^{A_R \bar{A}_R} + \bar{\omega}_a^{\bar{A}_R A_R}) \partial^a \end{aligned}$$

$$[D^{A_R}, \bar{D}^{\bar{A}_G}]_c = q[D'^{A_R}, \bar{D}^{\bar{A}_G}]_c = q[D^{A_R}, \bar{D}'^{\bar{A}_G}]_c = q^2[D'^{A_R}, \bar{D}'^{\bar{A}_G}]_c = Q^{A_R \bar{A}_G}$$

$$[D^{A_R}, \bar{D}^{\bar{A}_B}]_c = q[D'^{A_R}, \bar{D}^{\bar{A}_B}]_c = q[D^{A_R}, \bar{D}'^{\bar{A}_B}]_c = q^2[D'^{A_R}, \bar{D}'^{\bar{A}_B}]_c = Q^{A_R \bar{A}_B}$$

$$[D^{A_B}, D'^{A_R}]_c = -[D'^{A_B}, D^{A_R}]_c = (1-q)Q^{A_B A_R}$$

$$[D^{A_R}, \bar{D}^{\dot{\alpha}}]_c = q[D'^{A_R}, \bar{D}^{\dot{\alpha}}]_c = Q^{A_R \dot{\alpha}}$$

$$[Q^{AB}, \bar{Q}^{\bar{A}\bar{B}}] = -Q^{AB \bar{A}\bar{B}}, \quad [D^{\dot{\alpha}}, Q^{AB}] = -Q^{AB \dot{\alpha}}$$

$$[D^{A_R}, Q^{A_G A_B}]_c = [D'^{A_R}, Q^{A_G A_B}]_c = \rho_\beta^{A_R A_G A_B} D^\beta$$

$$[D^{A_R}, Q^{A_G \dot{\alpha}}]_c = q[D^{A_G}, Q^{A_R \dot{\alpha}}]_c = qQ^{A_R A_G \dot{\alpha}}$$

$$[D^{A_R}, Q^{A_G A_B \dot{\alpha}}]_c = [D'^{A_R}, Q^{A_G A_B \dot{\alpha}}]_c = \rho_\beta^{A_R A_G A_B} \sigma_b^{\beta \dot{\alpha}} \partial^b$$

$$[D^{A_R}, Q^{A_G A_B \bar{A}\bar{B}}]_c = [D'^{A_R}, Q^{A_G A_B \bar{A}\bar{B}}]_c = \rho_\beta^{A_R A_G B_B} Q^{\bar{A}\bar{B} \beta}$$

$$[\bar{D}^{\bar{A}_R}, Q^{A_R A_G \bar{A}_G \bar{A}_B}]_c = [\bar{D}'^{\bar{A}_R}, Q^{A_R A_G \bar{A}_G \bar{A}_B}]_c = q^{-1} \bar{\rho}_{\dot{\beta}}^{\bar{A}_R \bar{A}_G \bar{A}_B} Q^{A_R A_G \dot{\beta}}$$

$$[\bar{D}^{\bar{A}_R}, Q^{A_G A_B \bar{A}_G \bar{A}_B}]_c = [\bar{D}'^{\bar{A}_R}, Q^{A_G A_B \bar{A}_G \bar{A}_B}]_c = -\bar{\rho}_{\dot{\beta}}^{\bar{A}_R \bar{A}_G \bar{A}_B} Q^{A_G A_B \dot{\beta}}$$

$$[\bar{D}^{\bar{A}_R}, Q^{A_B A_R \bar{A}_G \bar{A}_B}]_c = [\bar{D}'^{\bar{A}_R}, Q^{A_B A_R \bar{A}_G \bar{A}_B}]_c = q \bar{\rho}_{\dot{\beta}}^{\bar{A}_R \bar{A}_G \bar{A}_B} Q^{A_B A_R \dot{\beta}}$$

$$[D^{A_R}, Q^{A_G A_B \bar{A}}]_c = [D'^{A_R}, Q^{A_G A_B \bar{A}}]_c = \rho_\beta^{A_R A_G B_B} Q^{\bar{A} \beta}$$

$$\begin{aligned}
[\bar{D}^{\bar{A}\bar{R}}, Q^{A_R A_G \bar{A}\bar{G}}]_c &= [\bar{D}'^{\bar{A}\bar{R}}, Q^{A_R A_G \bar{A}\bar{G}}]_c = Q^{A_R A_G \bar{A}\bar{R} \bar{A}\bar{G}} \\
[\bar{D}^{\bar{A}\bar{R}}, Q^{A_G A_B \bar{A}\bar{G}}]_c &= [\bar{D}'^{\bar{A}\bar{R}}, Q^{A_G A_B \bar{A}\bar{G}}]_c = -q Q^{A_G A_B \bar{A}\bar{R} \bar{A}\bar{G}} \\
[\bar{D}^{\bar{A}\bar{R}}, Q^{A_B A_R \bar{A}\bar{G}}]_c &= [\bar{D}'^{\bar{A}\bar{R}}, Q^{A_B A_R \bar{A}\bar{G}}]_c = q^2 Q^{A_B A_R \bar{A}\bar{R} \bar{A}\bar{G}} \\
[D^{A_R}, Q^{A_G \bar{A}}]_c &= [D'^{A_R}, Q^{A_G \bar{A}}]_c = q Q^{A_R A_G \bar{A}} \\
[\bar{D}^{\bar{A}\bar{R}}, Q^{A_R \bar{A}\bar{B}}]_c &= [\bar{D}'^{\bar{A}\bar{R}}, Q^{A_R \bar{A}\bar{B}}]_c = Q^{A_R \bar{A}\bar{B} \bar{A}\bar{R}} \\
[\bar{D}^{\bar{A}\bar{R}}, Q^{A_G \bar{A}\bar{B}}]_c &= [\bar{D}'^{\bar{A}\bar{R}}, Q^{A_G \bar{A}\bar{B}}]_c = -q^{-1} Q^{A_G \bar{A}\bar{B} \bar{A}\bar{R}} \\
[\bar{D}^{\bar{A}\bar{R}}, Q^{A_B \bar{A}\bar{B}}]_c &= [\bar{D}'^{\bar{A}\bar{R}}, Q^{A_B \bar{A}\bar{B}}]_c = -q Q^{A_B \bar{A}\bar{B} \bar{A}\bar{R}}
\end{aligned}$$

and of course all the similar relations obtained by simultaneously replacing R (resp. G , B , \bar{R} , \bar{G} , \bar{B}) by G (resp. B , R , \bar{G} , \bar{B} , \bar{R}) or by B (resp. R , G , \bar{B} , \bar{R} , \bar{G}) and/or by simultaneously conjugating all operators, ρ and $\bar{\rho}$. All other commutation relations give zero.

The generalized Jacobi identity is *always* satisfied:

$$\varepsilon(g_A, g_C)[[A, B]_c, C]_c + \varepsilon(g_B, g_A)[[B, C]_c, A]_c + \varepsilon(g_C, g_B)[[C, A]_c, B]_c = 0$$

for any A , B and C in the algebra. The main consequence of this is that it is isomorphic to an ordinary Lie superalgebra[1].

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